Cálculo de Probabilidades II – 2018-4
3º TVC

Questão A $E(X)=3,\ E(Y)=1,\ Var(X)=4,\ Var(Y)=9$
$Z = 5X - Y + 15$, X e Y independentes

(1)
$$\boxed{E(Z)} = 5E(X) - E(Y) + 15 = \boxed{29}$$
$$Var(Z) = 5^2 Var(X) + Var(Y) - 2\times 5\,\cancel{cov(X,Y)} \xrightarrow{} 0 \text{ (ind.)}$$
$$\boxed{Var(Z) = 109}$$

(2) $cov(X,Y) = 0{,}25$
$$E(Z) = 29$$
$$Var(Z) = 5^2 Var(X) + Var(Y) - 2\times 5 \times cov(X,Y).\sqrt{Var(X)Var(Y)}$$
$$= 109 - 10\times 0{,}25\sqrt{4\times 9}$$
$$\boxed{Var(Z) = 94}$$

(3) $E(X) = \mu,\ Var(X) = \sigma^2 \Rightarrow E(X^2) = \sigma^2 + \mu^2$
$$E(X-\mu)^3 = E(X^3 - 3X^2\mu + 3\mu^2 X - \mu^3)$$
$$= E(X^3) - 3\mu E(X^2) + 3\mu^2 E(X) - \mu^3$$
$$= E(X^3) - 3\mu(\sigma^2 + \cancel{\mu^2}) + \cancel{3\mu^3} - \mu^3$$
$$E(X-\mu)^3 = E(X^3) - 3\mu\sigma^2 - \mu^3$$

Questão B X_1 e X_2 independentes com fdp $f = \begin{cases} e^{-x}, & x>0; \\ 0, & cc \end{cases}$

$$f_{X_1X_2}(x_1,x_2) = e^{-x_1}.e^{-x_2} = e^{-(x_1+x_2)}$$

$W = \dfrac{X_1}{X_2} \Rightarrow X_2 = \dfrac{X_1}{W}$

$X_1 = X_1$

$$J(W,X_1) = \begin{vmatrix} \dfrac{\partial W}{\partial X_1} = \dfrac{1}{X_2} & \dfrac{\partial W}{\partial X_2} = -\dfrac{X_1}{X_2^2} \\ \dfrac{\partial X_1}{\partial X_1} = 1 & \dfrac{\partial X_1}{\partial X_2} = 0 \end{vmatrix}$$

$$J(W,X_1) = \cancel{\frac{X_1}{W^2}}\frac{X_1}{X_2^2} = \frac{W^2}{X_1}$$

(4) $$f_{WX_1}(w,x_1) = \frac{1}{\left|\frac{w^2}{x_1}\right|}\cdot f_{X_1}(x_1)\, f_{X_2}\left(\frac{x_1}{w}\right) = \frac{x_1}{w^2}e^{-x_1}e^{-\frac{x_1}{w}}$$

$$f_{WX_1}(w,x_1) = \frac{x_1}{w^2}e^{-x_1\left(1+\frac{1}{w}\right)}, \quad w>0,\ x_1>0$$

(5) $f_W(w) = \int_0^{\infty} \frac{x_1}{w^2} e^{-x_1(1+\frac{1}{w})} dx_1 = \frac{1}{w^2} \cdot \frac{1}{(1+\frac{1}{w})} \underbrace{\int_0^{\infty} x_1 \left(1+\frac{1}{w}\right) e^{-(1+\frac{1}{w})x_1} dx_1}_{E(\text{exp}(1+\frac{1}{w}))= \frac{1}{1+1/w}}$

$= \frac{1}{w^2} \cdot \frac{1}{(1+\frac{1}{w})} \times \frac{1}{1+\frac{1}{w}} = \begin{cases} \frac{1}{(1+w)^2}, & w>0 \\ 0, & cc \end{cases}$

$f_{X_1|W}(x_1|1) = \frac{\frac{x_1}{1^2} e^{-x_1(1+\frac{1}{1})}}{\frac{1}{(1+1)^2}} = \begin{cases} 4x_1 e^{-2x_1}, & x_1 \geq 0 \\ 0, & cc \end{cases}$

$X_1|W=1 \sim \text{gama}(\alpha=2, \lambda=2)$

6 - $P\{X_1 \geq 2 \mid W=1\} = \int_2^{\infty} 4x_1 e^{-2x_1} dx_1 = -2x_1 e^{-2x_1}\Big]_2^{\infty} - \int_2^{\infty} (-2e^{-2x_1}) dx_1$

$u = 2x_1 \quad du = 2dx_1$
$dv = 2e^{-2x_1} \quad v = -e^{-2x_1}$

$= 2\times 2 e^{-4} - \left[e^{-2x_1}\right]_2^{\infty} = 4e^{-4} + e^{-4} =$

$\boxed{P\{X_1 \geq 2 \mid W=1\} = 5e^{-4}}$

Questão e $\quad X \sim N(0,1)$ e $Y|X=x \sim N(2x-3, 12$
$(X,Y) \sim N_2(\underset{\sim}{\mu}, \Sigma)$

7. Marginal de Y: $\quad Y \sim N(\mu_Y, \sigma_Y^2)$

$\mu_Y = E(Y) = E[E(Y/X)] = E[2X-3] = \cancel{2E(X)}^{0} - 3$

$\boxed{\mu_Y = -3}$

$\sigma_Y^2 = Var(Y) = Var[E(Y/X)] + E[Var(Y/X)] = Var[2X-3] + E[12]$

$\sigma_Y^2 = 4\underbrace{Var(X)}_{1} + 12 = 4\times 1 + 12 \Rightarrow \boxed{\sigma_Y^2 = 16}$

$\boxed{Y \sim N(-3, 16)}$

⑧ $\operatorname{Corr}(X,Y) = \dfrac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}}$

$\operatorname{cov}(X,Y) = E(XY) - \cancel{E(X)}^{\,0}\cdot E(Y) = E\Big(E\big(X.Y/X\big)\Big)$

$= E\big(X\,E(Y/X)\big) = E\,(2X^2 - 3X) = 2\underbrace{E(X^2)}_{1} - \cancel{3E(X)}^{\,0}$

$\operatorname{Cov}(X,Y) = 2$

$\operatorname{Corr}(X,Y) = \dfrac{\operatorname{Cov}(X,Y)}{\sigma_X\,\sigma_Y} = \dfrac{2}{1\times 4} \Rightarrow \boxed{\operatorname{corr}(X,Y) = \frac{1}{2}}$

Questão D

$$F_X(x) = \begin{cases} 0, & x<0 \\ 1/4, & 0\le x<1 \\ 3/4, & 1\le x<2 \\ 1, & x\ge 2 \end{cases} \Rightarrow f_X(x) = \begin{cases} 0, & x<0 \\ 1/4, & x=0 \\ 1/2, & x=1 \\ 1/4, & x=2 \end{cases}$$

$E(X) = 0\times\frac{1}{4} + 1\times\frac{1}{2} + 2\times\frac{1}{4} \Rightarrow \boxed{E(X)=1}$

$\operatorname{Var}(X) = (0-1)^2\times\frac{1}{4} + \cancel{(1-1)^2}^{\,0}\times\frac{1}{2} + (2-1)^2\frac{1}{4} = \frac{1}{4}+\frac{1}{4} \Rightarrow \boxed{\operatorname{Var}(X) = 1/2}$

$S_n = \sum_{i=1}^{30} X_i \overset{\text{ind.}}{\Longrightarrow} \begin{array}{l} E(S_n) = 30\times 1 = 30 \\ \operatorname{Var}(S_n) = 30\times\frac{1}{2} = 15 \end{array}$

$P\{S_n \ge 33\} \approx P\left\{Z > \dfrac{32{,}5 - 30}{\sqrt{15}} = 0{,}646\right\} = 1 - \Phi(0{,}65)$

$\boxed{P\{S_n \ge 33\} = 1 - \Phi(0{,}65)}$