$$
\begin{aligned}
\Pr(W > M) &= \Pr(W - M > 0) \\
&= \Pr\left(Z > \frac{3}{5^{1/2}}\right) = \Pr(Z > 1.342) \\
&= 1 - \Phi(1.342) = 0.090.
\end{aligned}
$$

Thus, the probability that the woman will be taller than the man is 0.090. ◀

The Lognormal Distribution

It is very common to use normal distributions to model logarithms of random variables. For this reason, a name is given to the distribution of the original random variables before transforming. If $\log(X)$ has a normal distribution with mean μ and variance σ^2, we say that X has a lognormal distribution with parameters μ and σ^2.

Example 5.6.7 **Failure Times of Ball Bearings.** Products that are subject to wear and tear are generally tested for endurance in order to estimate their useful lifetimes. Lawless (1982, Example 5.2.2) describes data taken from Lieblein and Zelen (1956), which are measurements of the numbers of millions of revolutions before failure for 23 ball bearings. The lognormal distribuiton is one popular model for times until failure. Figure 5.4 shows a histogram of the 23 lifetimes together with a lognormal p.d.f. with parameters chosen to match the observed data. The bars of the histogram in Fig. 5.4 have a similar interpretation to those in Fig. 5.1, as described in Example 5.6.1. Suppose that the engineers are interested in knowing how long to wait until there is a 90 percent chance that a ball bearing will have failed. Then then they want the 0.9 quantile of the distribution of lifetimes. Let X be the time to failure of a ball bearing. The lognormal distribution of X plotted in Fig. 5.4 has parameters 4.15 and 0.5334^2. The d.f. of X would then be $F(x) = \Phi([\log(x) - 4.15]/0.5334)$, and the quantile function would be

$$F^{-1}(p) = e^{4.15+0.5334\Phi^{-1}(p)},$$

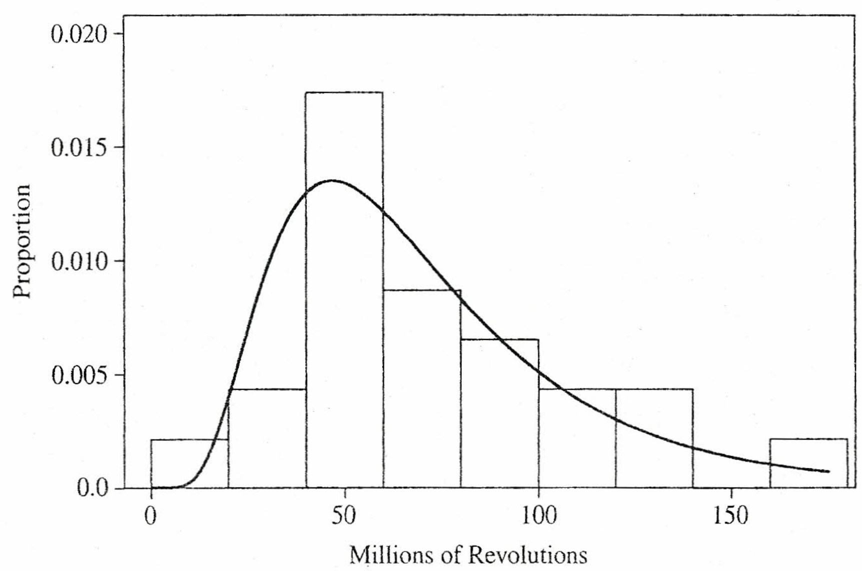

Figure 5.4 Histogram of lifetimes of ball bearings and fitted lognormal p.d.f. for Example 5.6.7.

where Φ^{-1} is the quantile function of the standard normal distribution. With $p = 0.9$, we get $\Phi^{-1}(0.9) = 1.28$ and $F^{-1}(0.9) = 125.6$. ◀

The moments of a lognormal random variable are easy to compute based on the m.g.f. of a normal distribution. If $Y = \log(X)$ has a normal distribution with mean μ and variance σ^2, then the m.g.f. of Y is $\psi(t) = \exp(\mu t + 0.5\sigma^2 t^2)$. But, the definition of ψ is $\psi(t) = E(e^{tY})$. Since $Y = \log(X)$, we have

$$\psi(t) = E(e^{tY}) = E(e^{t\log(X)}) = E(X^t).$$

It follows that $E(X^t) = \psi(t)$ for all real t. In particular, the mean and variance of X are

$$E(X) = \psi(1) = \exp(\mu + 0.5\sigma^2), \tag{5.6.9}$$
$$\mathrm{Var}(X) = \psi(2) - \psi(1)^2 = \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1].$$

Example 5.6.8 **Stock and Option Prices.** Consider a stock like the one in Example 5.6.2 whose current price is S_0. Suppose that the price at u time units in the future is $S_u = S_0 e^{Z_u}$, where Z_u has a normal distribution with mean μu and variance $\sigma^2 u$. Note that $S_0 e^{Z_u} = e^{Z_u + \log(S_0)}$ and $Z_u + \log(S_0)$ has a normal distribution with mean $\mu u + \log(S_0)$ and variance $\sigma^2 u$. So S_u has a lognormal distribution with parameters $\mu u + \log(S_0)$ and $\sigma^2 u$.

Black and Scholes (1973) developed a pricing scheme for options on stocks whose prices follow a lognormal distribution. For the remainder of this example, we shall consider a single time u and write the stock price as $S_u = S_0 e^{\mu u + \sigma u^{1/2} Z}$, where Z has a standard normal distribution. Suppose that we need to price the option to buy one share of the above stock for the price q at a particular time u in the future. As in Example 4.1.4 on page 186, we shall use risk-neutral pricing. That is, we force the present value of $E(S_u)$ to equal S_0. If u is measured in years and the risk-free interest rate is r per year, then the present value of $E(S_u)$ is $e^{-ru}E(S_u)$. (This assumes that compounding of interest is done continuously instead of just once as it was in Example 4.1.4. The effect of continuous compounding is examined in Exercise 25.) But $E(S_u) = S_0 e^{\mu u + \sigma^2 u/2}$. Setting S_0 equal to $e^{-ru} S_0 e^{\mu u + \sigma^2 u/2}$ yields $\mu = r - \sigma^2/2$ when doing risk-neutral pricing.

Now, we can determine a price for the specified option. The value of the option at time u will be $h(S_u)$, where

$$h(s) = \begin{cases} s - q & \text{if } s > q, \\ 0 & \text{otherwise.} \end{cases}$$

Now, set $\mu = r - \sigma^2/2$, and it is easy to see that $h(S_u) > 0$ if and only if

$$Z > \frac{\log\left(\frac{q}{S_0}\right) - (r - \sigma^2/2)u}{\sigma u^{1/2}}. \tag{5.6.10}$$

We shall refer to the constant on the right-hand side of Eq. (5.6.10) as c. The risk-neutral price of the option is the present value of $E(h(S_u))$, which equals

$$e^{-ru} E[h(S_u)] = e^{-ru} \int_c^{\infty} \left[S_0 e^{[r-\sigma^2/2]u + \sigma u^{1/2} z} - q \right] \frac{1}{(2\pi)^{1/2}} e^{-z^2/2}\, dz. \tag{5.6.11}$$

To compute the integral in Eq. (5.6.11), split the integrand into two parts at the $-q$. The second integral is then just a constant times the integral of a normal p.d.f., namely

$$-e^{-ru}q\int_c^\infty \frac{1}{(2\pi)^{1/2}}e^{-z^2}\,dz = -e^{-ru}q[1-\Phi(c)].$$

The first integral in Eq. (5.6.11), is

$$e^{-\sigma^2 u/2}S_0\int_c^\infty \frac{1}{(2\pi)^{1/2}}e^{-z^2/2+\sigma u^{1/2}z}\,dz.$$

This can be converted into the integral of a normal p.d.f. times a constant by completing the square (see Exercise 24). The result of completing the square is

$$e^{-\sigma^2 u/2}S_0\int_c^\infty \frac{1}{(2\pi)^{1/2}}e^{-(z-\sigma u^{1/2})^2/2+\sigma^2 u/2}\,dz = S_0[1-\Phi(c-\sigma u^{1/2})].$$

Finally, combine the two integrals into the option price, using the fact that $1-\Phi(x) = \Phi(-x)$:

$$S_0\Phi(\sigma u^{1/2}-c) - qe^{-ru}\Phi(-c). \tag{5.6.12}$$

This is the famous *Black-Scholes formula* for pricing options. As a simple example, suppose that $q = S_0$, $r = 0.06$ (six percent interest), $u = 1$ (one year wait), and $\sigma = 0.1$. Then (5.6.12) says that the option price should be $0.0746S_0$. If the distribution of S_u is different from the form used here, simulation techniques (see Chapter 11) can be used to help price options. ◀

The p.d.f. of the lognormal distribution will be found in Exercise 17 of this section. The d.f. of the lognormal distribution is easily constructed from the standard normal d.f. Φ. Let X have a lognormal distribution with parameters μ and σ^2. Then

$$\Pr(X \le x) = \Pr(\log(X) \le \log(x)) = \Phi\left(\frac{\log(x)-\mu}{\sigma}\right).$$

The results from earlier in this section about linear combinations of normal random variables translate into results about products of powers of lognormal random variables. Results about sums of independent normal random variables translate into results about products of independent lognormal random variables.

Summary

We introduced the family of normal distributions. The parameters of each normal distribution are its mean and variance. A linear combination of independent normal random variables has a normal distribution with mean equal to the linear combination of the means and variance determined by Corollary 4.3.1. In particular, if X has a normal distribution with mean μ and variance σ^2, then $(X-\mu)/\sigma$ has a standard normal distribution (mean 0 and variance 1). Probabilities and quantiles for normal distributions can be obtained from tables or computer programs for standard normal probabilities and quantiles. For example, if X has a normal distribution with mean μ and variance σ^2, then the d.f. of X is $F(x) = \Phi([x-\mu]/\sigma)$ and the quantile function of X is $F^{-1}(p) = \mu + \Phi^{-1}(p)\sigma$, where Φ is the standard normal d.f.

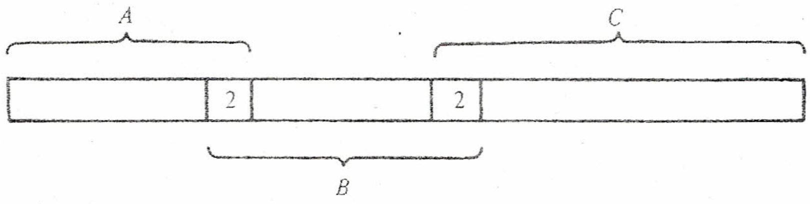

Figure 5.5 Sections of the rod in Exercise 9.

EXERCISES

1. Find the 0.5, 0.25, 0.75, 0.1, and 0.9 quantiles of the standard normal distribution.

2. Suppose that X has a normal distribution for which the mean is 1 and the variance is 4. Find the value of each of the following probabilities:

   a. $\Pr(X \le 3)$ b. $\Pr(X > 1.5)$
   c. $\Pr(X = 1)$ d. $\Pr(2 < X < 5)$
   e. $\Pr(X \ge 0)$ f. $\Pr(-1 < X < 0.5)$
   g. $\Pr(|X| \le 2)$ h. $\Pr(1 \le -2X + 3 \le 8)$.

3. If the temperature in degrees Fahrenheit at a certain location is normally distributed with a mean of 68 degrees and a standard deviation of 4 degrees, what is the distribution of the temperature in degrees Celsius at the same location?

4. Find the 0.25 and 0.75 quantiles of the Fahrenheit temperature at the location mentioned in Exercise 3.

5. Let $X_1, X_2,$ and X_3 be independent lifetimes of memory chips. Suppose that each X_i has a normal distribution with mean 300 hours and standard deviation 10 hours. Compute the probability that at least one of the three chips lasts at least 290 hours.

6. If the m.g.f. of a random variable X is $\psi(t) = e^{t^2}$ for $-\infty < t < \infty$, what is the distribution of X?

7. Suppose that the measured voltage in a certain electric circuit has a normal distribution with mean 120 and standard deviation 2. If three independent measurements of the voltage are made, what is the probability that all three measurements will lie between 116 and 118?

8. Evaluate the integral $\int_0^\infty e^{-3x^2}\,dx$.

9. A straight rod is formed by connecting three sections A, B, and C, each of which is manufactured on a different machine. The length of section A, in inches, has a normal distribution with mean 20 and variance 0.04. The length of section B, in inches, has a normal distribution with mean 14 and variance 0.01. The length of section C, in inches, has a normal distribution with mean 26 and variance 0.04. As indicated in Fig. 5.5, the three sections are joined so that there is an overlap of 2 inches at each connection. Suppose that the rod can be used in the construction of an airplane wing if its total length in inches is between 55.7 and 56.3. What is the probability that the rod can be used?

10. If a random sample of 25 observations is taken from a normal distribution with mean μ and standard deviation 2, what is the probability that the sample mean will lie within one unit of μ?

11. Suppose that a random sample of size n is to be taken from a normal distribution with mean μ and standard deviation 2. Determine the smallest value of n such that

$$\Pr(|\overline{X}_n - \mu| < 0.1) \ge 0.9.$$

12. a. Sketch the d.f. Φ of the standard normal distribution from the values given in the table at the end of this book.
    b. From the sketch given in part (a) of this exercise, sketch the d.f. of a normal distribution for which the mean is -2 and the standard deviation is 3.

13. Suppose that the diameters of the bolts in a large box follow a normal distribution with a mean of 2 centimeters and a standard deviation of 0.03 centimeter. Also, suppose that the diameters of the holes in the nuts in another large box follow a normal distribution with a mean of 2.02 centimeters and a standard deviation of 0.04 centimeter. A bolt and a nut will fit together if the diameter of the hole in the nut is greater than the diameter of the bolt and the difference between these diameters is not greater than 0.05 centimeter. If a bolt and a nut are selected at random, what is the probability that they will fit together?

14. Suppose that on a certain examination in advanced mathematics, students from university A achieve scores that are normally distributed with a mean of 625 and a variance of 100, and students from university B achieve scores which are normally distributed with a mean of 600 and a variance of 150. If two students from university A and three students from university B take this examination, what is the probability that the average of the scores of the two students from university A will be greater than the average of the scores of the three students from university B? *Hint:* Determine the distribution of the difference between the two averages.

15. Suppose that 10 percent of the people in a certain population have the eye disease glaucoma. For persons who have glaucoma, measurements of eye pressure X will be normally distributed with a mean of 25 and a variance of 1. For persons who do not have glaucoma, the pressure X will be normally distributed with a mean of 20 and a variance of 1. Suppose that a person is selected at random from the population and her eye pressure X is measured.
    a. Determine the conditional probability that the person has glaucoma given that $X = x$.
    b. For what values of x is the conditional probability in part (a) greater than 1/2?

16. Suppose that the joint p.d.f. of two random variables X and Y is

$$f(x, y) = \frac{1}{2\pi} e^{-(1/2)(x^2+y^2)} \quad \text{for } -\infty < x < \infty \text{ and } -\infty < y < \infty.$$

Find $\Pr(-\sqrt{2} < X + Y < 2\sqrt{2})$.

17. Consider a random variable X having a lognormal distribution with parameters μ and σ^2. Determine the p.d.f. of X.

18. Suppose that the random variables X and Y are independent and that each has a standard normal distribution. Show that the quotient X/Y has a Cauchy distribution.

19. Suppose that the measurement X of pressure made by a device in a particular system has a normal distribution with mean μ and variance 1, where μ is the true pressure. Suppose that the true pressure μ is unknown but has a uniform distribution on the interval [5, 15]. If $X = 8$ is observed, find the conditional p.d.f. of μ given $X = 8$.

20. Let X have a lognormal distribution with parameters 3 and 1.44. Find the probability that $X \le 6.05$.

21. Let X and Y be independent random variables such that $\log(X)$ has a normal distribution with mean 1.6 and variance 4.5 and $\log(Y)$ has a normal distribution with mean 3 and variance 6. Find the distribution of the product XY.

22. Suppose that X has a lognormal distribution with parameters μ and σ^2. Find the distribution of $1/X$.

23. Suppose that X has a lognormal distribution with parameters 4.1 and 8. Find the distribution of $3X^{1/2}$.

24. The method of *completing the square* is used several times in this text. It is a useful method for combining several quadratic and linear polynomials into a perfect square plus a constant. Prove the following identity, which is one general form of completing the square:

$$\sum_{i=1}^{n} a_i(x - b_i)^2 + cx =$$
$$\left(\sum_{i=1}^{n} a_i\right)\left(x - \frac{\sum_{i=1}^{n} a_i b_i - c/2}{\sum_{i=1}^{n} a_i}\right)^2$$
$$+ \sum_{i=1}^{n} a_i\left(b_i - \frac{\sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} a_i}\right)^2$$
$$+ \left(\sum_{i=1}^{n} a_i\right)^{-1}\left[c\sum_{i=1}^{n} a_i b_i - c^2/4\right].$$

if $\sum_{i=1}^{n} a_i \neq 0$.

25. In Example 5.6.8, we considered the effect of continuous compounding of interest. Suppose that S_0 dollars earn a rate of r per year componded continuously for u years. Prove that the principal plus interest at the end of this time equals $S_0 e^{ru}$. *Hint:* Suppose that interest is compounded n times at intervals of u/n years each. At the end of each of the n intervals, the principal gets multiplied by $1 + ru/n$. Take the limit of the result as $n \to \infty$.